2020 B

March 16

Orientation of a curve. whenever a parametrization of a curve is chosen,

$$\vec{r}(t), \quad t \in [a,b],$$

it has an orientation, that is, from $\vec{r}(a)$ to $\vec{r}(b)$.

The integral

$$\int_C f\, ds$$

is independent of the choice of the parametrization and also the orientation. However,

$$\int_C \vec{F} \cdot d\vec{r}$$

depends on the orientation. Indeed, when the orientation is reversed, the integral changes by a minus sign.

Let $\vec{r}(t)$, $t \in [a,b]$, be a regular curve. Its <u>unit tangent</u> at $\vec{r}(t)$ is

$$\hat{t} = \frac{x'(t)\,\hat{i} + y'(t)\,\hat{j}}{\sqrt{x'^2(t) + y'^2(t)}}$$

$$= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}.$$

($\vec{T}$ is used in Text). When the orientation of C is reversed, $\hat{t}$ become $-\hat{t}$. In the following we write $-C$ the curve of C with reverse orientation.

Claim:

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_{C} \vec{F}\cdot d\vec{r}$$

Pf of claim:

$$\int_{-C} \vec{F}\cdot d\vec{r} = \int_{-C} \vec{F}\cdot(-\hat{t})\,ds$$

$$= -\int_{-C} \vec{F}\cdot\hat{t}\,ds$$

$$= -\int_{C} \vec{F}\cdot\hat{t}\,ds$$

$$= -\int_{C} \vec{F}\cdot d\vec{r}.$$

$\vec{F}\cdot\hat{t}$ is a function and line integral of function is indent of any parametrization. Here C and $-C$ both are parametrization of the same curve.

We consider other meaning of $\int_C \vec{F}\cdot d\vec{r}$.

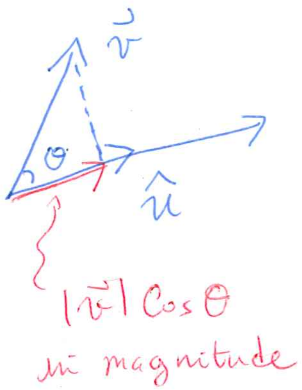
~ $\vec{F}$ is the force, $\int_C \vec{F}\cdot d\vec{r}$ is the work done of $\vec{F}$ along C.

~ $\vec{F}$ is the velocity of some fluid,
$\int_C \vec{F}\cdot d\vec{r}$ is the flow of $\vec{F}$ along C (the circulation of $\vec{F}$ around C when C is a closed curve.)

~ $\vec{F}$ is the velocity of some fluid, and $n=2$, C a simple closed curve,
$\int_C \vec{F}\cdot\hat{n}\,ds$ is the flux of $\vec{F}$ across C.

Let's explain the second meaning

Given vectors $\vec{u}$ and $\vec{v}$, when $|\vec{u}|=1$, $\vec{v}\cdot\vec{u}$ can be viewed as the projection of $\vec{v}$ onto the direction $\vec{u}$:



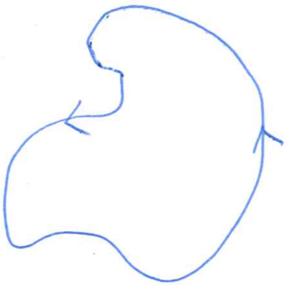
$$\vec{v}\cdot\hat{u} = |\vec{v}||\hat{u}|\cos\theta$$
$$= |\vec{v}|\cos\theta .$$

So $\vec{F}\cdot\hat{t}$ is the strength of $\vec{F}$ along the direction $\hat{t}$,

$$\int_C \vec{F}\cdot\hat{t}\, ds = \int_C \vec{F}\cdot d\vec{r}$$

is the amount of flow along C in unit time.

Next, the third meaning.



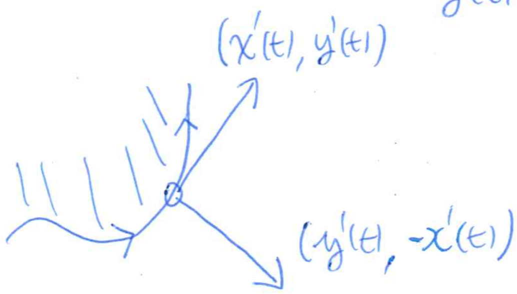
Let C be a simple closed curve in anticlockwise direction (Counterclockwise, positive direction)

$\vec{r}(t)$, $t\in[a,b]$, $\vec{r}(a)=\vec{r}(b)$.

$\vec{r}'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j}$ points to the tangent direction

Then

$y'(t)\,\hat{i} + (-x'(t))\,\hat{j}$ points to the outer normal direction.



Hence,

$\int_C \vec{F}\cdot\hat{n}\, ds$ measure the amount of flow across C (going out of C)

Let $\vec{F} = M\hat{i} + N\hat{j}$ and $\vec{r}(t)$ parametric C.

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_a^b \vec{F}(x(t),y(t))\cdot(y'(t)\hat{i} - x'(t)\hat{j})\,dt$$

$$= \int_a^b M(x(t),y(t))y'(t) - N(x(t),y(t))x'(t)\,dt$$

$$= \int_a^b (-Nx' + My')\,dt$$

$$= \int_a^b M\,dy - N\,dx,$$

which is the flux in terms of M, N.

e.g. Find the work done of $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$ along the helix

$$\vec{r}(t) = (\cos t, \sin t, t), \quad t \in [0, \pi/2].$$

(Or, find the flow of $\vec{F}$ along $\vec{r}(t)$.)

$$\vec{r}'(t) = (-\sin t, \cos t, 1)$$

$$\vec{F}(\vec{r}(t)) = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$$

$$\therefore \quad \vec{F}(r(t))\cdot\vec{r}'(t) = -\sin t\cos t + t\cos t + \sin t$$

$$W.D = \int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\,dt$$

$$= \frac{\pi}{2} - \frac{1}{2}. \#$$

× × ×

We study some special v.f's.

Let $\vec{F}$ be a smooth v.f. in open region Ω.

It is a <u>gradient</u> v.f. (or a <u>conservative</u> v.f.) if

$$\vec{F} = \nabla f, \qquad \text{some function } f.$$

f is called the <u>potential</u> of $\vec{F}$.

e.g. the gravitational field

$$\vec{F} = -gMm \frac{\vec{r}}{|\vec{r}|^3}, \qquad \vec{r} \neq \vec{0}.$$

has potential

$$f = \frac{gMm}{|\vec{r}|} = \frac{gMm}{\sqrt{x^2+y^2+z^2}}.$$

<u>Theorem</u> $\vec{F} = \nabla f$. Then for any curve C from A to B,

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A).$$

In particular, when C is a closed curve,

$$\oint_C \vec{F}\cdot d\vec{r} = 0.$$

A, B are posotion of the particle at $t=a$, and $t=b$, they are vectors.

Pf: $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$

$$= \int_a^b \left(M(\vec{r}(t))x'(t) + N(\vec{r}(t))y'(t) + P(\vec{r}(t))z'(t)\right)dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}(\vec{r}(t))x'(t) + \frac{\partial f}{\partial y}(\vec{r}(t))y'(t) + \frac{\partial f}{\partial z}(\vec{r}(t))z'(t)\right)dt$$

$$= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t))\,dt$$

$$= f(x(b), y(b), z(t)) - f(x(a), y(a), z(a))$$

$$= f(B) - f(A).$$

Here, by the Chain Rule,

$$\frac{d}{dt} f(x(t), y(t), z(t)) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) + \frac{\partial f}{\partial z}z'(t). \quad \#$$